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A Family of Generalized Jacobi Polynomials

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Dedicated to Professor Günther Hämmerlin on his sixtieth birthday

Abstract. The family of orthogonal polynomials corresponding to a generalized Jacobi weight function was considered by Wheeler and Gautschi who derived recurrence relations, both for the related Chebyshev moments and for the associated orthogonal polynomials. We obtain an explicit representation of these polynomials, from which the recurrence relation can be derived.

1. Introduction. The family of normalized orthogonal polynomials $P_\mu(\cdot;\lambda)$ corresponding to the weight function

$$\omega^{(\alpha,\beta,\gamma)}(x;\lambda) := \begin{cases} |x|^{2\gamma+1}(1-x^2)^\alpha(x^2-\lambda^2)^\beta, & \lambda \le |x| \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$\gamma \in \mathbf{R}$, $\alpha > -1$, $\beta > -1$, $0 < \lambda < 1$, has been considered by Barkov [1], Gautschi [3] and in the special case $\gamma = 0$, $\alpha = \beta = \pm\frac{1}{2}$ by Wheeler [6]. This generalized Chebyshev case is of some interest in theoretical chemistry. Wheeler showed that the related Chebyshev moments of $\omega^{(-1/2,-1/2,0)}(\cdot;\lambda)$ as functions of λ^2 may be computed recursively. We pointed out that this recursion follows from the fact that these moments essentially are orthogonal polynomials up to a linear factor [4]. Wheeler and Gautschi were primarily interested in the recurrence relation of the orthogonal polynomials $P_\mu(\cdot;\lambda)$. This recursion formula was derived in case $\gamma = 0$ and for general Jacobi parameters $\alpha, \beta > -1$; special attention was given to the Chebyshev case $\alpha = \beta = \pm\frac{1}{2}$ [3]. Our aim is to derive an explicit representation of the orthogonal polynomials in the Jacobi case $\gamma = 0$, $\alpha, \beta > -1$ and in some other cases where γ is an even integer. Finally, we obtain the coefficients of the recurrence relation by using the known coefficients of the Jacobi recursion. These coefficients were derived implicitly by Gautschi [3] who gave recursions for them.

2. Reduction to Jacobi-Like Form. In a first step we reduce some integrals with weight function $\omega^{(\alpha,\beta,\gamma)}$ to a generalized Jacobi form. We set

$$v := \frac{2}{1-\lambda^2}\left(x^2 - \frac{1+\lambda^2}{2}\right)$$

or

$$x^2 = K(v+\rho), \quad K := \frac{1-\lambda^2}{2}, \quad \rho := \frac{1+\lambda^2}{1-\lambda^2}.$$

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Then we get by substitution

$$\begin{aligned}
\int_{-1}^{1} f(x^2)\omega^{(\alpha,\beta,\gamma)}(x;\lambda)\,dx &= 2\int_{\lambda}^{1} f(x^2)x^{2\gamma+1}(1-x^2)^{\alpha}(x^2-\lambda^2)^{\beta}\,dx \\
&= K^{\alpha+\beta+\gamma+1}\int_{-1}^{1} f(K(v+\rho))(v+\rho)^{\gamma}(1-v)^{\alpha}(1+v)^{\beta}\,dv \qquad (2.1)\\
&= K^{\alpha+\beta+\gamma+1}\int_{-1}^{1} f(K(v+\rho))\mu^{(\alpha,\beta,\gamma)}(v;\rho)\,dv,
\end{aligned}$$

where $\mu^{(\alpha,\beta,\gamma)}$ denotes the weight function

$$\mu^{(\alpha,\beta,\gamma)}(v;\rho) = \begin{cases} (1-v)^{\alpha}(1+v)^{\beta}(\rho+v)^{\gamma}, & |v| \le 1, \\ 0, & \text{elsewhere}, \end{cases} \qquad (2.2)$$

$\gamma \in \mathbf{R}$, $\alpha > -1$, $\beta > -1, \rho > 1$. The case $\rho < -1$ may be solved by replacing v by $-v$ and interchanging α and β. We see that the integrals (2.1) are of Jacobi type if $\gamma = 0$. They reduce to another special case if $\gamma = s \in \mathbf{Z}$. Then the weight function μ is the product of the Jacobi weight with a polynomial resp. a rational function with s-fold zero resp. pole at $v = -\rho$, $\rho > 1$. We will show that in these cases the orthogonal polynomials may be represented in terms of Jacobi polynomials.

As the weight $\omega^{(\alpha,\beta,\gamma)}(x;\lambda)$ is an even function of x, the associated orthogonal polynomials $P_n^{(\alpha,\beta,\gamma)}(x;\lambda)$ of even and odd degree n are even and odd functions, respectively. So we have

$$P_{2n}^{(\alpha,\beta,\gamma)}(x;\lambda) = \varphi_n^{(\alpha,\beta,\gamma)}(x^2;\lambda), \qquad P_{2n+1}^{(\alpha,\beta,\gamma)}(x;\lambda) = x\psi_n^{(\alpha,\beta,\gamma)}(x^2;\lambda),$$

with polynomials φ_n and ψ_n of exact degree n. (In the sequel we omit the parameters $\alpha, \beta, \gamma, \lambda$ if possible.)

In the *even* degree case, the orthonormality relation becomes, in view of (2.1),

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1} P_{2n}(x)P_{2m}(x)\omega^{(\alpha,\beta,\gamma)}(x;\lambda)\,dx \\
&= \int_{-1}^{1} \varphi_n(x^2)\varphi_m(x^2)\omega^{(\alpha,\beta,\gamma)}(x;\lambda)\,dx \qquad (2.3)\\
&= K^{\alpha+\beta+\gamma+1}\int_{-1}^{1} \varphi_n(K(v+\rho))\varphi_m(K(v+\rho))\mu^{(\alpha,\beta,\gamma)}(v;\rho)\,dv.
\end{aligned}$$

In the *odd* degree case we get similarly

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1} P_{2n+1}(x)P_{2m+1}(x)\omega^{(\alpha,\beta,\gamma)}(x;\lambda)\,dx \\
&= \int_{-1}^{1} \psi_n(x^2)\psi_m(x^2)x^2\omega^{(\alpha,\beta,\gamma)}(x;\lambda)\,dx \qquad (2.4)\\
&= K^{\alpha+\beta+\gamma+2}\int_{-1}^{1} \psi_n(K(v+\rho))\psi_m(K(v+\rho))\mu^{(\alpha,\beta,\gamma+1)}(v;\rho)\,dv.
\end{aligned}$$

Let in the usual notation of Jacobi polynomials [5]

$$h_n^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}.$$

Then the orthonormality relations (2.3) and (2.4) are fulfilled in case $\gamma = 0$ if

$$\varphi_n(K(v+\rho)) = [K^{\alpha+\beta+1} h_n^{(\alpha,\beta)}]^{-1/2} P_n^{(\alpha,\beta)}(v) \tag{2.5}$$

and in case $\gamma = -1$ if

$$\psi_n(K(v+\rho)) = [K^{\alpha+\beta+1} h_n^{(\alpha,\beta)}]^{-1/2} P_n^{(\alpha,\beta)}(v). \tag{2.6}$$

If we replace $K(v+\rho)$ by x^2, there follows

PROPOSITION 2.1. *In case $\gamma = 0$, the even-degree orthonormal polynomials have the form*

$$\begin{aligned} &P_{2n}^{(\alpha,\beta,0)}(x;\lambda) \\ &\quad = \left[\left(\frac{1-\lambda^2}{2}\right)^{\alpha+\beta+1} h_n^{(\alpha,\beta)}\right]^{-1/2} P_n^{(\alpha,\beta)}\left(\frac{2}{1-\lambda^2}x^2 - \frac{1+\lambda^2}{1-\lambda^2}\right), \end{aligned} \tag{2.7}$$

and in case $\gamma = -1$, the odd-degree orthonormal polynomials have the form

$$\begin{aligned} &P_{2n+1}^{(\alpha,\beta,-1)}(x;\lambda) \\ &\quad = \left[\left(\frac{1-\lambda^2}{2}\right)^{\alpha+\beta+1} h_n^{(\alpha,\beta)}\right]^{-1/2} x P_n^{(\alpha,\beta)}\left(\frac{2}{1-\lambda^2}x^2 - \frac{1+\lambda^2}{1-\lambda^2}\right). \end{aligned} \tag{2.8}$$

3. The Case $\gamma = s, s \in \mathbf{N}$. If the parameter γ is a natural number, the weight function $\mu^{(\alpha,\beta,\gamma)}$ has the special form

$$\mu^{(\alpha,\beta,s)}(v;\rho) = (1-v)^\alpha(1+v)^\beta(\rho+v)^s,$$

where $\rho > 1$, $s \in \mathbf{N}$, $\alpha, \beta > -1$. Next to the Jacobi case $\gamma = s = 0$ we now consider the case $s = 1$; then other values of $s \in \mathbf{N}$ can be treated by induction.

According to an idea of Christoffel (cf. Szegö [5, p. 29 ff.]) we define the sequence of *monic* polynomials

$$p_n^{(\alpha,\beta,1)}(v;\rho) := \frac{1}{P_n^{(\alpha,\beta)}(\sigma)k_{n+1}^{(\alpha,\beta)}} \frac{P_n^{(\alpha,\beta)}(\sigma)P_{n+1}^{(\alpha,\beta)}(v) - P_{n+1}^{(\alpha,\beta)}(\sigma)P_n^{(\alpha,\beta)}(v)}{v-\sigma}, \tag{3.1}$$

where $\sigma := -\rho$, and in the usual notation

$$k_n^{(\alpha,\beta)} := \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}. \tag{3.2}$$

By direct inspection—numerator and denominator are both zero at $v = \sigma$—or via the Christoffel-Darboux identity it is easy to see that $p_n^{(\alpha,\beta,1)}$ is a monic polynomial of degree n. For every polynomial q_{n-1} of degree less than n we get

$$\begin{aligned} &\int_{-1}^{1} p_n^{(\alpha,\beta,1)}(v;\rho) q_{n-1}(v)\mu^{(\alpha,\beta,1)}(v;\rho)\,dv \\ &\quad = \frac{1}{k_{n+1}^{(\alpha,\beta)}}\int_{-1}^{1}\left\{P_{n+1}^{(\alpha,\beta)}(v) - \frac{P_{n+1}^{(\alpha,\beta)}(\sigma)}{P_n^{(\alpha,\beta)}(\sigma)}P_n^{(\alpha,\beta)}(v)\right\} \\ &\qquad\qquad \times q_{n-1}(v)(1-v)^\alpha(1+v)^\beta\,dv \\ &\quad = 0 \end{aligned} \tag{3.3}$$

because of the orthogonality of the Jacobi polynomials. Thus, $p_n^{(\alpha,\beta,1)}(v;\rho)$ is a constant multiple of the orthonormal polynomial relative to the weight $\mu^{(\alpha,\beta,1)}$.

The normalization can be done with the help of the Christoffel-Darboux identity (Szegö [5, p. 42 ff.]), from which it follows that

$$p_n^{(\alpha,\beta,1)}(v;\rho) = \frac{h_n^{(\alpha,\beta)}}{k_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\sigma)} \sum_{\nu=0}^{n} [h_\nu^{(\alpha,\beta)}]^{-1} P_\nu^{(\alpha,\beta)}(\sigma) P_\nu^{(\alpha,\beta)}(v). \tag{3.4}$$

By writing one factor $p_n^{(\alpha,\beta,1)}$ in the fractional form (3.1) and the other in the Christoffel-Darboux sum form (3.4), we get

$$\begin{aligned} &\int_{-1}^{1} [p_n^{(\alpha,\beta,1)}(v;\rho)]^2 \mu^{(\alpha,\beta,1)}(v;\rho)\,dv \\ &= \frac{h_n^{(\alpha,\beta)}}{k_{n+1}^{(\alpha,\beta)} k_n^{(\alpha,\beta)} [P_n^{(\alpha,\beta)}(\sigma)]^2} \sum_{\nu=0}^{n} [h_\nu^{(\alpha,\beta)}]^{-1} P_\nu^{(\alpha,\beta)}(\sigma) \\ &\quad \times \int_{-1}^{1} P_\nu^{(\alpha,\beta)}(v) [P_n^{(\alpha,\beta)}(\sigma) P_{n+1}^{(\alpha,\beta)}(v) - P_{n+1}^{(\alpha,\beta)}(\sigma) P_n^{(\alpha,\beta)}(v)] \\ &\qquad\qquad \times (1-v)^\alpha (1+v)^\beta\,dv \\ &= -\frac{h_n^{(\alpha,\beta)}}{k_{n+1}^{(\alpha,\beta)} k_n^{(\alpha,\beta)}} \frac{P_{n+1}^{(\alpha,\beta)}(-\rho)}{P_n^{(\alpha,\beta)}(-\rho)} = \frac{h_n^{(\alpha,\beta)}}{k_{n+1}^{(\alpha,\beta)} k_n^{(\alpha,\beta)}} \frac{P_{n+1}^{(\beta,\alpha)}(\rho)}{P_n^{(\beta,\alpha)}(\rho)}. \end{aligned} \tag{3.5}$$

We thus have

PROPOSITION 3.1. *The polynomials*

$$q_n^{(\alpha,\beta,1)}(v;\rho) := \frac{1}{\sqrt{h_n^{(\alpha,\beta,1)}}} \frac{P_n^{(\beta,\alpha)}(\rho) P_{n+1}^{(\alpha,\beta)}(v) + P_{n+1}^{(\beta,\alpha)}(\rho) P_n^{(\alpha,\beta)}(v)}{\rho + v},$$

$$h_n^{(\alpha,\beta,1)} := h_n^{(\alpha,\beta)} k_{n+1}^{(\alpha,\beta)} [k_n^{(\alpha,\beta)}]^{-1} P_n^{(\beta,\alpha)}(\rho) P_{n+1}^{(\beta,\alpha)}(\rho)$$

are orthonormal with respect to $(1-v)^\alpha(1+v)^\beta(\rho+v)$. *An alternative representation is*

$$q_n^{(\alpha,\beta,1)}(v;\rho) = \frac{(-1)^n h_n^{(\alpha,\beta)}}{\sqrt{h_n^{(\alpha,\beta,1)}}} k_{n+1}^{(\alpha,\beta)} [k_n^{(\alpha,\beta)}]^{-1} \sum_{\nu=0}^{n} [h_\nu^{(\alpha,\beta)}]^{-1} P_\nu^{(\alpha,\beta)}(-\rho) P_\nu^{(\alpha,\beta)}(v).$$

From (2.4) it follows that the polynomials ψ_n for the *odd*-degree orthogonal polynomials have the representation

$$\psi_n^{(\alpha,\beta,0)}(v) = [K^{\alpha+\beta+2}]^{-1/2} q_n^{(\alpha,\beta,1)}(K^{-1}v - \rho;\rho). \tag{3.6}$$

Thus we get

PROPOSITION 3.2. *In case* $\gamma = 0$ *the odd-degree orthonormal polynomials have the form*

$$P_{2n+1}^{(\alpha,\beta,0)}(x;\lambda) = \left(\frac{1-\lambda^2}{2}\right)^{-(\alpha+\beta+2)/2} x q_n^{(\alpha,\beta,1)}\left(\frac{2}{1-\lambda^2}x^2 - \frac{1+\lambda^2}{1-\lambda^2}; \frac{1+\lambda^2}{1-\lambda^2}\right).$$

4. The Recurrence Relation. In case $\gamma = 0$ we now derive the recursion formula for the even and odd degree polynomials, respectively. We start with a known result from the theory of orthogonal polynomials (Chihara [2, p. 25]).

LEMMA 4.1. *If the system $\{P_n\}_{n=0}^{\infty}$ of monic polynomials can be generated by the recurrence relation*

$$\begin{aligned} &P_0(x)=1, \qquad P_1(x)=(x-\beta_0)P_0(x), \\ &P_{n+1}(x)=(x-\beta_n)P_n(x)-\gamma_n P_{n-1}(x), \qquad n=1,2,\dots, \end{aligned} \tag{4.1}$$

then for $a,b\in\mathbf{R}$, $a\neq 0$, the system $\{Q_n\}_{n=0}^{\infty}$ of monic polynomials

$$Q_n(x):=a^{-n}P_n(ax+b)$$

can be generated by

$$\begin{aligned} &Q_0(x)=1, \qquad Q_1(x)=(x-\beta_0')Q_0(x), \\ &Q_{n+1}(x)=(x-\beta_n')Q_n(x)-\gamma_n' Q_{n-1}(x), \qquad n=1,2,\dots, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \beta_n' &:= \frac{\beta_n-b}{a}, \qquad n=0,1,2,\dots, \\ \gamma_n' &:= \frac{\gamma_n}{a^2}, \qquad n=1,2,\dots. \end{aligned} \tag{4.3}$$

From (2.5) there follows the representation of the *monic even*-degree polynomials $\tilde{\varphi}_n^{(\alpha,\beta,0)}$:

$$\tilde{\varphi}_n^{(\alpha,\beta,0)}(v)=K^n[k_n^{(\alpha,\beta)}]^{-1}P_n^{(\alpha,\beta)}\left(\frac{v}{K}-\rho\right). \tag{4.4}$$

We are interested in the coefficients b_{2n}, c_{2n} of the recursion formula

$$\tilde{\varphi}_{n+1}^{(\alpha,\beta,0)}(v)=(v-b_{2n})\tilde{\varphi}_n^{(\alpha,\beta,0)}(v)-c_{2n}\tilde{\varphi}_{n-1}^{(\alpha,\beta,0)}(v). \tag{4.5}$$

Now we use the recursion of the *monic* Jacobi polynomials (Chihara [2, p. 220])

$$\begin{aligned} \tilde{P}_{n+1}^{(\alpha,\beta)}(x) &= (x-\beta_n^{(\alpha,\beta)})\tilde{P}_n^{(\alpha,\beta)}(x)-\gamma_n^{(\alpha,\beta)}\tilde{P}_{n-1}^{(\alpha,\beta)}(x), \\ \beta_n^{(\alpha,\beta)} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\ \gamma_n^{(\alpha,\beta)} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{4.6}$$

and take $a=1/K$, $b=-\rho$, $b/a=-(1+\lambda^2)/2$ in Lemma 4.1 to obtain

$$\begin{aligned} b_{2n} &= \frac{1+\lambda^2}{2}+K\beta_n^{(\alpha,\beta)} \\ &= \frac{1+\lambda^2}{2} \qquad \text{if } |\alpha|=|\beta|,\ n=0,1,2,\dots, \end{aligned} \tag{4.7}$$

$$\begin{aligned} c_{2n} &= \left(\frac{1-\lambda^2}{2}\right)^2\gamma_n^{(\alpha,\beta)} \\ &= \left(\frac{1-\lambda^2}{4}\right)^2 \qquad \text{if } \alpha=\beta=\pm\tfrac{1}{2}, \qquad n=1,2,\dots. \end{aligned} \tag{4.8}$$

To get the recursion formula for the *monic odd*-degree polynomials in case $\gamma=0$ we start from Proposition 3.1 and use Lemma 4.1. (Alternatively, we could start with Proposition 3.2, but the following computation in two steps is easier.) We know that the monic polynomials

$$\tilde{q}_n^{(\alpha,\beta,1)}(v):=\frac{P_n^{(\alpha,\beta)}(-\rho)P_{n+1}^{(\alpha,\beta)}(v)-P_{n+1}^{(\alpha,\beta)}(-\rho)P_n^{(\alpha,\beta)}(v)}{k_{n+1}^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(-\rho)(v+\rho)} \tag{4.9}$$

are orthogonal with respect to $(1-v)^\alpha(1+v)^\beta(v+\rho)$. The coefficients σ_n, τ_n in the recursion formula

$$\tilde{q}_{n+1}^{(\alpha,\beta,1)}(v) = (v-\sigma_n)\tilde{q}_n^{(\alpha,\beta,1)}(v) - \tau_n \tilde{q}_{n-1}^{(\alpha,\beta,1)}(v) \tag{4.10}$$

result from

$$\begin{aligned} &\frac{\tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)\tilde{P}_{n+2}^{(\alpha,\beta)}(v) - \tilde{P}_{n+2}^{(\alpha,\beta)}(-\rho)\tilde{P}_{n+1}^{(\alpha,\beta)}(v)}{\tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)} \\ &\quad = (v-\sigma_n)\frac{\tilde{P}_n^{(\alpha,\beta)}(-\rho)\tilde{P}_{n+1}^{(\alpha,\beta)}(v) - \tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)\tilde{P}_n^{(\alpha,\beta)}(v)}{\tilde{P}_n^{(\alpha,\beta)}(-\rho)} \\ &\qquad - \tau_n \frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(-\rho)\tilde{P}_n^{(\alpha,\beta)}(v) - \tilde{P}_n^{(\alpha,\beta)}(-\rho)\tilde{P}_{n-1}^{(\alpha,\beta)}(v)}{\tilde{P}_{n-1}^{(\alpha,\beta)}(-\rho)}, \end{aligned} \tag{4.11}$$

if we first multiply by $\rho + v$. Applying to the terms involving the factor v the recurrence formula (4.6) (with $x = v$) results in a vanishing linear combination of Jacobi polynomials. Equating the coefficients to zero then gives

$$\sigma_n = \beta_{n+1}^{(\alpha,\beta)} + \frac{\tilde{P}_{n+2}^{(\alpha,\beta)}(-\rho)}{\tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)} - \frac{\tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)}{\tilde{P}_n^{(\alpha,\beta)}(-\rho)}, \tag{4.12}$$

$$\tau_n = \frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(-\rho)\tilde{P}_{n+1}^{(\alpha,\beta)}(-\rho)}{[\tilde{P}_n^{(\alpha,\beta)}(-\rho)]^2}\gamma_n^{(\alpha,\beta)}. \tag{4.13}$$

As the monic odd-degree polynomial $\tilde{\psi}_n^{(\alpha,\beta,0)}$ has the representation (cf. (3.6))

$$\tilde{\psi}_n^{(\alpha,\beta,0)}(v) = K^n q_n^{(\alpha,\beta,1)}\left(\frac{v}{K} - \rho\right), \tag{4.14}$$

the coefficients b_{2n+1}, c_{2n+1} in the formula

$$\tilde{\psi}_{n+1}^{(\alpha,\beta,0)}(v) = (v - b_{2n+1})\tilde{\psi}_n^{(\alpha,\beta,0)}(v) - c_{2n+1}\tilde{\psi}_{n-1}^{(\alpha,\beta,0)}(v) \tag{4.15}$$

can be derived from (4.12) and (4.13) using Lemma 4.1. Thus we get

$$b_{2n+1} = \frac{1+\lambda^2}{2} + \frac{1-\lambda^2}{2}\sigma_n, \tag{4.16}$$

$$c_{2n+1} = \left(\frac{1-\lambda^2}{2}\right)^2 \tau_n. \tag{4.17}$$

In the Chebyshev case $\alpha, \beta = \pm\frac{1}{2}$ these formulas can be simplified. We set

$$\varsigma := -\rho - \sqrt{\rho^2 - 1}; \tag{4.18}$$

then the Chebyshev polynomial of the first kind can be represented in the form

$$T_n(-\rho) = \tfrac{1}{2}(\varsigma^n + \varsigma^{-n}) \tag{4.19}$$

and therefore

$$\frac{\tilde{P}_{n+2}^{(-1/2,-1/2)}(-\rho)}{\tilde{P}_{n+1}^{(-1/2,-1/2)}(-\rho)} = \frac{1}{2}\frac{\varsigma^{n+2} + \varsigma^{-n-2}}{\varsigma^{n+1} + \varsigma^{-n-1}}. \tag{4.20}$$

As in this case

$$\beta_{n+1} = 0, \qquad \gamma_{n+1} = \tfrac{1}{4},$$

we get for $\alpha = \beta = -\frac{1}{2}$

$$b_{2n+1} = \frac{1+\lambda^2}{2} + \frac{1-\lambda^2}{4} \cdot \frac{\varsigma^2 - 2 + \varsigma^{-2}}{(\varsigma^{n+1} + \varsigma^{-n-1})(\varsigma^n + \varsigma^{-n})}, \tag{4.21}$$

$$c_{2n+1} = \left(\frac{1-\lambda^2}{4}\right)^2 \left[1 - \frac{\varsigma^2 - 2 + \varsigma^{-2}}{(\varsigma^n + \varsigma^{-n})^2}\right]. \tag{4.22}$$

We note that in the limit $\lambda \to 0$, i.e., $\varsigma \to 1$ or $\rho \to 1$, these recursion coefficients agree with those of the polynomial $P_n^{(-1/2,1/2)}(2v-1)$ up to a scaling. Analogous formulas can be derived in the other Chebyshev cases.

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